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Equality of immanantal decomposable tensors

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Abstract

We state a necessary and sufficient condition for equality of nonzero decomposable symmetrized tensors when the symmetrizer is associated to an irreducible character of the symmetric group of degree m, corresponding to a partition of m with the form $(p, p, \dots, p)$.

Keywords: Decomposable symmetrized tensors; Matroid colorings

1. Introduction

The concept of tensor product has been extended to the notion of symmetry class of tensors to deal with the symmetry in multilinear mappings. The most familiar symmetry classes of tensors are the exterior powers of a vector space V $(\wedge^m V, m \in \mathbb{N})$ and the symmetric powers of V, $(\vee^m V,\ m \in \mathbb{N})$. The concepts and the terminology used to study the tensor product of vector spaces apply to this new setting in a natural way. So extensions of the classical questions on tensor products follow from this generalization. One of these questions seeks relations between two ordered families of nonzero vectors $(x_1, \dots, x_m)$ and $(y_1, \dots, y_m)$ when the equality

$$x_1 \otimes \cdots \otimes x_m = y_1 \otimes \cdots \otimes y_m \tag{1}$$

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holds. These relations are well known and they consist of the proportionality of homologous factors of the left- and right-hand side of (1) (i.e. $x_i = c_i y_i$ for every i) and a normalization condition on the proportionality coefficients ($\prod_{i=1}^m c_i = 1$).

The answer to the corresponding questions for the mth exterior power and the mth symmetric power is also well known. In the case of the mth exterior power (the problem only makes sense if $(x_1, \dots, x_m)$ and $(y_1, \dots, y_m)$ are linearly independent) the equality

$$x_1 \wedge \cdots \wedge x_m = y_1 \wedge \cdots \wedge y_m$$

holds if and only if the families $(x_1, \dots, x_m)$ and $(y_1, \dots, y_m)$ have the same linear closure, i.e.

$$\langle x_i \ : \ i = 1, \dots, m\rangle = \langle y_i \ : \ i = 1, \dots, m\rangle$$

and the normalization condition $\det(a_{ij}) = 1$ holds, where $y_i = \sum_{j=1}^m a_{ij} x_j$, $i = 1, \dots, m$. In the case of the mth symmetric power the equality

$$x_1 \vee \cdots \vee x_m = y_1 \vee \cdots \vee y_m$$

holds if and only if factors of the right-hand side are proportional to the factors of the left-hand side i.e. there exists a permutation $(i_1, \dots, i_m)$ of the integers $(1, \dots, m)$ such that $x_j = c_j y_{i_j}$, $j = 1, \dots, m$ and the normalization property $\prod_{i=1}^m c_i = 1$ is satisfied.

The mth exterior power and the mth symmetric power both are special cases of symmetry classes related to a symmetry connected with the irreducible characters of S_m. We will call these symmetry classes, immanantal symmetry classes of tensors.

In this paper we present a necessary and sufficient condition for the equality of immanantal decomposable tensors with symmetry associated with irreducible characters of S_m of the form $(p, \dots, p)$. It is worth remarking that the exterior powers and the symmetric powers of V are immanantal symmetry classes associated with irreducible characters of this form. Moreover the conditions we present "mix" the above conditions for the exterior powers and the symmetric powers.

2. Preliminaries

Let S_m be the symmetric group of degree m and let $\lambda = (\lambda_1, \dots, \lambda_t)$, $\lambda_1 \geqslant \cdots \geqslant \lambda_t > 0$ be a partition of m. The integer t (the greatest integer indexing a positive coordinate of λ) is called the length of λ. Let $\mu = (\mu_1, \dots, \mu_r)$ be a partition of m. We say that λ majorizes μ and denote $\lambda \succeq \mu$, if $t \leqslant r$ and

$$\sum_{i=1}^{j} \lambda_i \geqslant \sum_{i=1}^{j} \mu_i, \quad j = 1, \dots, t.$$

The sequence $\lambda' = (\lambda'_1, \dots, \lambda'_{\lambda_1})$, where

$$\lambda'_i = |\{j \ : \ \lambda_j \geqslant i\}|,$$

is a partition of m, called conjugate partition of λ.

Throughout this article, when X is a set $|X|$ denotes the cardinality of X, and Id_X denotes the identity mapping from X into X.

We denote by $[\lambda]$ the Young diagram corresponding to λ. In this paper we identify the boxes of $[\lambda]$ with integers of $\{1, \ldots, m\}$ number the m boxes of $[\lambda]$ from left to right from top to bottom.

A *filling* D of the diagram $[\lambda]$ with the integers $1, \ldots, m$ such that each integer occurs once in the filling will be called a strict Young tableau. To each permutation $\sigma \in S_m$, there corresponds a strict Young tableau $D_{\lambda,\sigma}$ in which box i of $[\lambda]$ is filled with the integer $\sigma(i)$. If $D = D_{\lambda,\sigma}$ is a strict Young tableau and $\nu \in S_m$, the strict Young tableau $D_{\lambda,\nu\sigma}$ is denoted by νD. We say that a strict Young tableau D is increasing by columns (by rows) if the integers in each column (each row) of D are in increasing order. We say that a strict Young tableau is standard if it is simultaneously increasing by rows and by columns.

Given a strict Young tableau D, the subgroup of S_m of the permutations ν such that D and νD have the same rows is called the group of rows of D and is denoted by $R(D)$ (briefly by R if there are no ambiguities to avoid). Similarly we define the group $C(D)$ (denoted briefly by C) of columns of D.

Let $\mathbb{C}$ be the complex field. Let V be a finite dimensional vector space over $\mathbb{C}$. Let $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ be linearly independent families of vectors of V that span the same subspace. If $y_i = \sum_{j=1}^{m} a_{ij} x_j$, $i = 1, \ldots, m$, $(a_{ij} \in \mathbb{C})$ the $m \times m$ matrix $[a_{ij}]$ is denoted by

$$M[y_1, \ldots, y_m | x_1, \ldots, x_m].$$

Let $\Gamma = \{i_1, \ldots, i_s\}$, $(i_1 < \cdots < i_s)$ and $\Delta = \{j_1, \ldots, j_s\}$, $(j_1 < \cdots < j_s)$ be subsets of $\{1, \ldots, m\}$ of cardinality s. Assume that $(x_i)_{i \in \Gamma}$ is linearly independent and $\langle x_i : i \in \Gamma \rangle = \langle y_i : i \in \Delta \rangle$. We denote the $s \times s$ matrix $M[y_{j_1}, \ldots, y_{j_s} | x_{i_1}, \ldots, x_{i_s}]$ by

$$M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta|\Gamma].$$

If $\Gamma = \Delta$ and if $\langle x_i : i \in \Delta \rangle = \langle y_i : i \in \Delta \rangle$, we denote the $s \times s$ matrix $M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta|\Delta]$ by

$$M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta].$$

Definition 2.1. Let $(x_1, \ldots, x_m)$ be a family of nonzero vectors of V. We say that a collection $C = (C_1, \ldots, C_r)$, $(C_i = \{x_j : j \in \Delta_i\})_{i=1,\ldots,r}$ of subfamilies of $(x_1, \ldots, x_m)$, is a coloring of $(x_1, \ldots, x_m)$ if the following conditions hold:

(1) C_i is linearly independent, $i = 1, \ldots, r$;
(2) $\Delta_i \cap \Delta_j = \emptyset$, $i \neq j$, $i, j = 1, \ldots, r$;
(3) $\bigcup_{i=1}^{r} \Delta_j = \{1, \ldots, m\}$;
(4) $|\Delta_1| \geqslant \cdots \geqslant |\Delta_r|$.

The collection $(\Delta_1, \dots, \Delta_r)$ is called the support of the coloring $C = (C_1, \dots, C_r)$. The finite sequence $(|\Delta_1|, \dots, |\Delta_r|)$ is a partition of m called shape of the coloring C and denoted by

$$\mathrm{shape}(C).$$

If $\mathrm{shape}(C) = \mu$ we also say that C is a μ-coloring of $(x_1, \dots, x_m)$.

Let $(x_1, \dots, x_m)$ be a family of nonzero vectors of V. In [3] it was proved that, with respect to majorization, the set of the shapes of colorings of $(x_1, \dots, x_m)$ as a maximum. This maximum partition is the rank partition of $(x_1, \dots, x_m)$ and we denote by $\rho(x_1, \dots, x_m)$, i.e.

$$\rho(x_1, \dots, x_m) = \max_{\succeq}\{\mathrm{shape}(C) : C \text{ is a coloring of } (x_1, \dots, x_m)\}.$$

A ρ-coloring of $(x_1, \dots, x_m)$, where ρ is the rank partition of $(x_1, \dots, x_m)$, is called a factorization of $(x_1, \dots, x_m)$.

Denote by $\Gamma_{m,n}$ the set of the maps from $\{1, 2, \dots, m\}$ into $\{1, 2, \dots, n\}$ and denote by $Q_{m,n}$ the subset of $\Gamma_{m,n}$ of the strictly increasing maps. If $A = [a_{ij}]$ is a $m \times n$ matrix and if $\alpha \in \Gamma_{p,m}$ and $\beta \in \Gamma_{q,n}$, denote by $A[\alpha|\beta]$ the $p \times q$ matrix whose (i, j)-entry is $a_{\alpha(i)\beta(j)}$.

If V is an inner product vector space and W is a subspace of V, we denote by $W^\perp$ the orthogonal complement of W. We denote by $\otimes^m V$ the mth tensor power of V. If V is an inner product space we consider in $\otimes^m V$ the inner product defined by the equalities

$$\begin{aligned}(u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_m) \\ = \prod_{i=1}^{m}(u_i, v_i), \quad u_i, v_i \in V,\ i = 1, \dots, m.\end{aligned} \tag{2}$$

For $\sigma \in S_m$ let $P(\sigma)$ be the unique linear operator of $\otimes^m V$ satisfying for all $u_1, \dots, u_m \in V$,

$$P(\sigma)(u_1 \otimes \cdots \otimes u_m) = u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(m)}.$$

An operator in the linear closure of $\{P(\sigma)|\sigma \in S_m\}$ is called a symmetrizer and its range is called symmetry class of tensors. The image of $u_1 \otimes \cdots \otimes u_m$ by a symmetrizer is called a decomposable symmetrized tensor and is denoted by

$$u_1 * \cdots * u_m.$$

Let D be a strict Young tableau. Denote by $P(D)$ the symmetrizer

$$P(D) = \sum_{\sigma \in R(D)} P(\sigma)$$

and by $N(D)$ we denote the symmetrizer

$$N(D) = \sum_{\sigma \in C(D)} \epsilon(\sigma) P(\sigma).$$

The symmetrizer $E(D) = P(D)N(D)$ is the Young symmetrizer associated with D.

Theorem 2.1 [6]. *Let μ be a partition of m and let D be a Young diagram associated with μ. Let $x_1, \dots, x_m \in V$. Then*

$$E(D)(x_1 \otimes \cdots \otimes x_m) \neq 0$$

if and only if

$$N(D)(x_1 \otimes \cdots \otimes x_m) \neq 0.$$

If H is a subgroup of S_m and λ is an irreducible character of H, the symmetrizer

$$T(H, \lambda) := \frac{\lambda(id)}{|H|} \sum_{\sigma \in H} \lambda(\sigma) P(\sigma),$$

is an orthogonal projection (see [11]), and its range is denoted by $V_\lambda^m(H)$.

Observe that

$$P(D) = |R(D)| T(R(D), 1) \tag{3}$$

and

$$N(D) = |C(D)| T(C(D), \varepsilon). \tag{4}$$

We call the symmetry classes $V_\lambda^m(S_m)$, immanantal symmetry class of tensors and denote $V_\lambda^m(S_m)$ by V_λ^m. The decomposable symmetrized tensors in V_λ^m are called immanantal decomposable tensors or decomposable elements of V_λ^m. If $\lambda = \epsilon$ (the alternating character), we have $V_\epsilon^m = \wedge^m V$, and if $\lambda = 1$ we have $V_1^m = \vee^m V$.

Let $\Delta = \{i_1, \dots, i_s\} \quad (i_1 < \cdots < i_s)$ be a subset of $\{1, \dots, m\}$. The decomposable element $x_{i_1} \wedge \cdots \wedge x_{i_s}$ will be denoted by

$$\bigwedge_{i \in \Delta} x_i.$$

The main result of this paper is a necessary and sufficient condition for equality of immanantal decomposable tensors, in symmetry classes whose corresponding irreducible character of S_m is associated to a partition of m of the form $(p, \dots, p)$.

The next result presents a splitting of the symmetrizer $T(S_m, \lambda)$ into pairwise orthogonal idempotent symmetrizers.

If $\sigma, \tau \in S_m$, we say that $\sigma > \tau$ if

$$(\sigma(1), \dots, \sigma(m)) > (\tau(1), \dots, \tau(m))$$

by the lexicographic order. Based in a result of [8, p. 78], the following result is proved in [4]:

Proposition 2.1. *Let λ be a partition of m. Let $id = \sigma_1, \dots, \sigma_d$ be permutations of S_m satisfying*:

(1) $\sigma_i < \sigma_j$ *if* $i < j$, $i, j = 1, \dots, d$;
(2) $D_{\lambda,\sigma_1}, \dots, D_{\lambda,\sigma_d}$ *are the standard strict Young tableaux associated with the Young diagram* $[\lambda]$.

Let $\nu \in S_m$. *Let* $D = \nu D_{\lambda,id}$. *Define* $P_i = P(\nu D_{\lambda,\sigma_i})$ *and* $N_i = N(\nu D_{\lambda,\sigma_i})$, $i = 1, \dots, d$. *Let*

$$e_{i,i} = \frac{\lambda(id)}{m!} M_i P_i N_i, \quad i = 1, \dots, d,$$

where $M_1 = Id_{\otimes^m V}$ *and* $M_i = Id_{\otimes^m V} - e_{1,1} - e_{2,2} - \cdots - e_{i-1,i-1}$, $i = 2, \dots, d$. *Then,*

$$T(S_m, \lambda) = e_{1,1} + \cdots + e_{d,d}.$$

Moreover, $e_{1,1}, \dots, e_{d,d}$ *are pairwise orthogonal idempotents.*

Let Ψ be a subset of $\{1, \dots, m\}$. We denote by S_Ψ the subgroup of S_m of the permutations that fix each element in the complement of Ψ. Let $\mu = (\mu_1, \dots, \mu_r)$ be a partition of m and let $D = D_{\mu,\sigma}$ be a strict Young tableau. Let $k_{i,j}$ be the integer that fills the box in row i and column j of $D_{\mu,\sigma}$. Let θ be the permutation of S_m defined by the equalities

$$\theta(\mu_1 + \cdots + \mu_{j-1} + t) = k_{j,t},$$

$j = 1, \dots, r$ and $t = 1, \dots, \mu_j$. When we use as a model of the mth tensor power of V the pair $(\gamma, \otimes^m V)$, where $\gamma = P(\theta^{-1}) \circ \otimes$, we denote $\gamma(x_1, \dots, x_m)$ by $x_1 \widetilde{\otimes} \cdots \widetilde{\otimes} x_m$, or by $x_1 \otimes \cdots \otimes x_m$ if there are no ambiguities to avoid. Let $\Psi_i = \{k_{i,1}, \dots, k_{i,\mu_i}\}$, $(k_{i,1} < \cdots < k_{i,\mu_i})$, $i = 1, \dots, r$. It can be easily checked that if $H = S_{\Psi_1} \times \cdots \times S_{\Psi_r}$ and if $\lambda = \epsilon$,

$$T(H, \lambda)(x_1 \widetilde{\otimes} \cdots \widetilde{\otimes} x_m) = \left(\bigwedge_{i \in \Psi_1} x_i \right) \otimes \cdots \otimes \left(\bigwedge_{i \in \Psi_r} x_i \right)$$

for all $x_1, \dots, x_m \in V$.

Since we will work with nonzero immanantal decomposable tensors, it is useful to say something about the vanishing of these tensors. In [2], Gamas obtained a necessary and sufficient condition for a immanantal decomposable tensor to be zero. The following theorem appeared in [3] and is a reformulation of that result.

Theorem 2.2. *Consider the symmetry class of tensors associated with the irreducible character* λ *of* S_m. *The immanantal decomposable tensor*

$$x_1 * \cdots * x_m$$

is different of zero if and only if there exist a λ'*-coloring of the family* $x_1, \dots, x_m$; *if and only if* $\rho(x_1, \dots, x_m) \succeq \lambda'$.

3. Grassmannians

Let V be a n-dimensional vector space over $\mathbb{C}$. This section is devoted to the study of decomposable tensors of $\wedge^k V$. It is well known that $v_1 \wedge \cdots \wedge v_k \neq 0$ if and only if $(v_1, \ldots, v_k)$ is linearly independent. It is also well known that if $\sigma \in S_k$ then

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \epsilon(\sigma) v_1 \wedge \cdots \wedge v_k.$$

Given a basis $\{e_1, \ldots, e_n\}$ of V, the set

$$\{e_\alpha^\wedge \ : \ \alpha \in Q_{k,n}\},$$

where $e_\alpha^\wedge$ denotes the decomposable element $e_{\alpha(1)} \wedge \cdots \wedge e_{\alpha(k)}$, is a basis of $\wedge^k V$.

The next result gives a necessary and sufficient condition for an element $z \in \wedge^k V$ to be decomposable.

Theorem 3.1 [10]. *Let V be an n-dimensional vector space over $\mathbb{C}$ and let $\{e_1, \ldots, e_n\}$ be a basis of V. Let*

$$z = \sum_{\alpha \in Q_{k,n}} a_\alpha e_\alpha^\wedge$$

be an element of $\wedge^k V$. Then z is decomposable if and only if exists a $k \times n$ matrix A over $\mathbb{C}$ such that

$$a_\alpha = \det(A[\eta|\alpha]), \quad \alpha \in Q_{k,n},$$

where η denotes the map $(1, \ldots, k)$.

Denote by $\mathscr{G}_{k,n}(V)$, or simply by $\mathscr{G}$, the set of all decomposable tensors of $\wedge^k V$. The set $\mathscr{G}$ is an affine algebraic variety and the family of Plucker's Polynomials is a system of defining polynomials for this variety [12, p. 45].

From now on we will refer to affine algebraic varieties shortly as algebraic varieties.

Proposition 3.1 [7, p. 15]. *Let $f_1, \ldots, f_n \in \mathbb{C}[T_1, \ldots, T_m]$, where $T_1, \ldots, T_m$ are independent variables, and let U_0 be the subset of $\mathbb{C}^n$,*

$$U_0 = \{(f_1(t_1, \ldots, t_m), \ldots, f_n(t_1, \ldots, t_m)) : (t_1, \ldots, t_m) \in \mathbb{C}^m\}.$$

The closure of U_0 in the Zariski topology is an irreducible algebraic variety.

A subset $\mathscr{U}$ of $\mathbb{C}^n$ satisfying the conditions U_0 fulfills in Proposition 3.1 is called parametrizable.

Corollary 3.1. *Let $\mathscr{U}$ be an algebraic variety. If $\mathscr{U}$ is parameterizable then $\mathscr{U}$ is irreducible.*

Using this result it follows from Theorem 3.1 that $\mathscr{G}$ is an irreducible algebraic variety.

The next theorem gives a necessary and sufficient condition for orthogonality of decomposable elements of $\wedge^k V$.

Theorem 3.2 [1]. *Let $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ be families of linearly independent vectors of the inner product vector space V. Then, $x_1 \wedge \cdots \wedge x_k$ is orthogonal to $y_1 \wedge \cdots \wedge y_k$ if and only if*

$$\langle x_1, \ldots, x_k\rangle^{\perp} \cap \langle y_1, \ldots, y_k\rangle \neq \{0\}.$$

The following result shows that the intersection of the grassmmanian $\mathscr{G}_{k,n}(V)$ with a hyperplane of $\wedge^k V$ whose axis is decomposable is again irreducible.

Theorem 3.3. *Let $(x_1, \ldots, x_k)$ be a linearly independent family of vectors of V. Let $\mathscr{S}$ be the orthogonal complement of the one-dimensional vector space spanned by a nonzero decomposable element, $x_1 \wedge \cdots \wedge x_k$, of $\wedge^k V$ i.e.,*

$$\mathscr{S} = \langle x_1 \wedge \cdots \wedge x_k\rangle^{\perp}.$$

Then $\mathscr{G} \cap \mathscr{S}$ is an irreducible algebraic variety.

Proof. We prove that $\mathscr{G} \cap \mathscr{S}$ is parametrizable. Let $(e_1, \ldots, e_n)$ be an orthogonal basis of V satisfying

$$\langle e_1, \ldots, e_k\rangle = \langle x_1, \ldots, x_k\rangle.$$

Since $(e^{\wedge}_{\alpha})_{\alpha \in Q_{k,n}}$ is an orthogonal basis of $\wedge^k V$ (see [9]), we have

$$\langle x_1 \wedge \cdots \wedge x_k\rangle^{\perp} = \langle e^{\wedge}_{\alpha} : \alpha \in \mathfrak{A}\rangle,$$

where $\mathfrak{A} = Q_{k,n} - \{\eta\}$, $\eta = (1, \ldots, k)$.

Denote by T and $\mathbb{T}$ respectively the following scalar matrix and matrix of indeterminates:

$$T := \begin{pmatrix} 0 & \cdots & 0 & t_{1k+1} & \cdots & t_{1n} \\ t_{21} & \cdots & t_{2k} & t_{2k+1} & \cdots & t_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{k1} & \cdots & t_{kk} & t_{kk+1} & \cdots & t_{kn} \end{pmatrix},$$

$$\mathbb{T} := \begin{pmatrix} 0 & \cdots & 0 & T_{1k+1} & \cdots & T_{1n} \\ T_{21} & \cdots & T_{2k} & T_{2k+1} & \cdots & T_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{k1} & \cdots & T_{kk} & T_{kk+1} & \cdots & T_{kn} \end{pmatrix}.$$

Consider the map

$$\Phi_1 : \mathbb{C}^{n-k} \times \mathbb{C}^n \times \cdots \times \mathbb{C}^n \longrightarrow \mathscr{G} \cap \mathscr{S},$$

$$(t_{1k+1}, \ldots, t_{1n}, \ldots, t_{k1}, \ldots, t_{kn}) \longrightarrow \sum_{\alpha \in \mathfrak{A}} \det(T[\eta|\alpha]) e^{\wedge}_{\alpha}.$$

Is easy to see that Φ_1 is well defined. For each $\alpha \in \mathfrak{A}$ denote by $\mathfrak{h}_\alpha$ the polynomial $\det(\mathbb{T}[\eta|\alpha])$ belonging to $\mathbb{C}[T_{1k+1}, \dots, T_{1n}, \dots, T_{k1}, \dots, T_{kn}]$, and identify the elements of $\wedge^k V$ with their $\binom{n}{k}$-tuple of coordinates in the basis $(e^\wedge_\alpha)_{\alpha \in Q_{k,n}}$. Then the range of Φ_1 is

$$\operatorname{Im} \Phi_1 = \Big\{ (\mathfrak{h}_\alpha\, (t_{1k+1}, \dots, t_{1n}, \dots, t_{k1}, \dots, t_{kn}))_{\alpha \in \mathfrak{A}}\, , \\ (t_{1k+1}, \dots, t_{1n}, \dots, t_{k1}, \dots, t_{kn}) \in \mathbb{C}^{kn-k} \Big\}.$$

If we prove that Φ_1 is onto we can conclude, by Corollary 3.1, that $\mathscr{G} \cap \mathscr{S}$ is an irreducible algebraic variety.

Let $(c_\alpha)_{\alpha \in \mathfrak{A}} \in \mathscr{G} \cap \mathscr{S}$. Then, $z = \sum_{\alpha \in \mathfrak{A}} c_\alpha e^\wedge_\alpha$ is a decomposable tensor of $\wedge^k V$ and z belongs to $\langle x_1 \wedge \cdots \wedge x_k \rangle^\perp$. Therefore, there exist $v_1, \dots, v_k \in V$ such that

$$z = v_1 \wedge \cdots \wedge v_k.$$

Using Theorem 3.2 and the conditions for equality of decomposable elements of $\wedge^k V$, we can assume, without loss of generality, that $v_1 \in \langle x_1, \dots, x_k \rangle^\perp$. Then,

$$v_1 = \sum_{j=k+1}^{n} a_{1j} e_j \quad \text{and} \quad v_i = \sum_{j=1}^{n} a_{ij} e_j, \quad i = 2, \dots, k$$

and so,

$$z = \sum_{\alpha \in \mathfrak{A}} \det(A[\eta|\alpha]) e^\wedge_\alpha,$$

where

$$A = \begin{pmatrix} 0 & \cdots & 0 & a_{1k+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2k} & a_{2k+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} & a_{kk+1} & \cdots & a_{kn} \end{pmatrix}.$$

Let

$$(a_{1k+1}, \dots, a_{1n}, \dots, a_{k1}, \dots, a_{kn}) \in \mathbb{C}^{kn-k}.$$

We obtain

$$\Phi_1((a_{1k+1}, \dots, a_{1n}, \dots, a_{k1}, \dots, a_{kn})) = z$$

and the proof is complete. □

4. Equality of immanantal decomposable tensors

This paper addresses conditions for equality of immanantal decomposable tensors. If $\lambda = \epsilon$ or $\lambda = 1$, these conditions have been known for some time [11] as discussed in Section 1.

If λ is not a linear character of S_m the problem of finding conditions for equality of immanantal decomposable tensors is not solved. However some partial results have appeared in recent years. The main theorem proved in [4] is a necessary and sufficient condition for equality of decomposable tensors of V_λ^m with the assumption that the rank partition of $(x_1, \dots, x_m)$ and $(y_1, \dots, y_m)$ is λ'.

Theorem 4.1 [4]. *Let* $(x_1, \dots, x_m)$ *and* $(y_1, \dots, y_m)$ *be families of nonzero vectors of* V *with rank partition* λ'. *Then,*

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m$$

if and only if the following conditions hold:

(1) *The set of supports of the factorizations of* $(x_1, \dots, x_m)$ *is equal to the set of supports of the factorizations of* $(y_1, \dots, y_m)$;
(2) *If* $(\Delta_1, \dots, \Delta_{\lambda_1})$ *is a support of a factorization of* $(x_1, \dots, x_m)$ *then*

$$\langle x_i : i \in \Delta_j \rangle = \langle y_i : i \in \Delta_j \rangle, \quad j = 1, \dots, \lambda_1$$

and

$$\prod_{i=1}^{\lambda_1} \det M[y_1, \dots, y_m | x_1, \dots, x_m][\Delta_i] = 1.$$

A necessary condition for equality of immanantal decomposable tensors, unconstrained by the families of vectors $(x_1, \dots, x_m)$ and $(y_1, \dots, y_m)$, was given in [5].

Theorem 4.2 [5]. *Let* λ *be an irreducible character of* S_m *and let* $(x_1, \dots, x_m)$ *and* $(y_1, \dots, y_m)$ *be families of nonzero vectors of* V. *If*

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m \neq 0,$$

then every support of a λ'*-coloring of* $(x_1, \dots, x_m)$ *is also a support of a* λ'*-coloring of* $(y_1, \dots, y_m)$.

Next theorem is the main result of this paper.

Theorem 4.3. *Let* $\lambda = (p, \dots, p)$. *Let* $(x_1, \dots, x_m)$ *and* $(y_1, \dots, y_m)$ *be families of nonzero vectors of* V. *Then,*

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m \neq 0 \tag{5}$$

if and only if the following conditions hold:

(1) *The set of supports of the* λ'*-colorings of* $(x_1, \dots, x_m)$ *is equal to the set of supports of the* λ'*-colorings of* $(y_1, \dots, y_m)$;
(2) *If* $(\Delta_1, \dots, \Delta_p)$ *is a support of a* λ'*-coloring of* $(x_1, \dots, x_m)$ *then, there exists* $\sigma \in S_p$ *such that*

$$\langle x_i : i \in \Delta_j \rangle = \langle y_i : i \in \Delta_{\sigma(j)} \rangle, \quad j = 1, \dots, p$$

and

$$\prod_{i=1}^{p} \det M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta_{\sigma(i)} | \Delta_i] = 1.$$

Proof. First we prove the only if part.

Let $\Delta_1 = \{i_1, \ldots, i_k\}$, $(i_1 < \cdots < i_k)$. Let $(e_1, \ldots, e_n)$ be an ordered orthogonal basis of V satisfying

$$\langle e_1, \ldots, e_k \rangle = \langle x_{i_1}, \ldots, x_{i_k} \rangle = \langle x_i \ : \ i \in \Delta_1 \rangle.$$

As we have remarked before, since $(e_\alpha^\wedge)_{\alpha \in Q_{k,n}}$ is an orthogonal basis of $\wedge^k V$,

$$\left\langle \bigwedge_{i \in \Delta_1} x_i \right\rangle^{\perp} = \langle e_\alpha^\wedge : \alpha \in \mathfrak{A} \rangle,$$

where $\mathfrak{A} = Q_{k,n} - \{\eta\}$, $\eta = (1, \ldots, k)$. We denote by $\overline{e}_\alpha^\wedge$ the unit tensor

$$\overline{e}_\alpha^\wedge := \frac{1}{\|e_\alpha^\wedge\|} e_\alpha^\wedge.$$

Assume that the length of λ is k. If $\dim V = k$ it follows from equality (5) and Theorem 2.2 that

$$\rho(x_1, \ldots, x_m) = \rho(y_1, \ldots, y_m) = \lambda'.$$

Now using Theorem 4.1 we get the only if part (choosing $\sigma = \mathrm{id} \in S_p$).

Assume now that $\dim V > k$. By Theorem 4.2 the first condition is satisfied so we only have to prove the second condition. This we do by induction.

Let $(\Delta_1, \ldots, \Delta_p)$ be the support of a λ'-coloring of $(x_1, \ldots, x_m)$ and let D be a strict Young tableau associated with $[\lambda]$, whose ith column is Δ_i, $i = 1, \ldots, p$. Then, by Theorem 2.1,

$$\begin{aligned} E(D)(x_1 \otimes \cdots \otimes x_m) &= \frac{m!}{\lambda(\mathrm{id})} e_{11}(x_1 * \cdots * x_m) \\ &= \frac{m!}{\lambda(\mathrm{id})} e_{11}(y_1 * \cdots * y_m) \\ &= E(D)(y_1 \otimes \cdots \otimes y_m) \neq 0. \end{aligned} \tag{6}$$

Let $u_1, \ldots, u_k$ be a linearly independent family of vectors of V and let $u'_1, \ldots, u'_m$ be a family of vectors taken from $u_1, \ldots, u_k$ by choosing $u'_j = u_i$ if j belongs to the ith row of D.

By (6), we get

$$\begin{aligned} &(u'_1 \otimes \cdots \otimes u'_m, E(D)(x_1 \otimes \cdots \otimes x_m)) \\ &\quad = (u'_1 \otimes \cdots \otimes u'_m, E(D)(y_1 \otimes \cdots \otimes y_m)). \end{aligned} \tag{7}$$

Since $T(R, 1)$ is an orthogonal projection, using equalities (3) and (4), we obtain from (7) that

$$\begin{aligned}&(T(R,1)(u'_1 \otimes \cdots \otimes u'_m), T(C,\epsilon)(x_1 \otimes \cdots \otimes x_m))\\ &\quad = (T(R,1)(u'_1 \otimes \cdots \otimes u'_m), T(C,\epsilon)(y_1 \otimes \cdots \otimes y_m)).\end{aligned}$$

Bearing in mind the way we have chosen the vectors $u'_1, \ldots, u'_m$, we can see that for all $\sigma \in R$ we have

$$P(\sigma)(u'_1 \otimes \cdots \otimes u'_m) = u'_1 \otimes \cdots \otimes u'_m.$$

So

$$T(R,1)(u'_1 \otimes \cdots \otimes u'_m) = u'_1 \otimes \cdots \otimes u'_m.$$

Since $T(C, \epsilon)$ is an orthogonal projection we obtain

$$\begin{aligned}&(T(C,\epsilon)(u'_1 \otimes \cdots \otimes u'_m), T(C,\epsilon)(x_1 \otimes \cdots \otimes x_m))\\ &\quad = (T(C,\epsilon)(u'_1 \otimes \cdots \otimes u'_m), T(C,\epsilon)(y_1 \otimes \ldots \otimes y_m)).\end{aligned}$$

Then

$$\begin{aligned}&\left(\left(\bigwedge_{i=1}^{k} u_i\right) \otimes \cdots \otimes \left(\bigwedge_{i=1}^{k} u_i\right), \left(\bigwedge_{i\in\Delta_1} x_i\right) \otimes \cdots \otimes \left(\bigwedge_{i\in\Delta_p} x_i\right)\right)\\ &\quad = \left(\left(\bigwedge_{i=1}^{k} u_i\right) \otimes \cdots \otimes \left(\bigwedge_{i=1}^{k} u_i\right), \left(\bigwedge_{i\in\Delta_1} y_i\right) \otimes \cdots \otimes \left(\bigwedge_{i\in\Delta_p} y_i\right)\right).\end{aligned}$$

Therefore

$$\prod_{j=1}^{p}\left(\bigwedge_{i=1}^{k} u_i, \bigwedge_{i\in\Delta_j} x_i\right) = \prod_{j=1}^{p}\left(\bigwedge_{i=1}^{k} u_i, \bigwedge_{i\in\Delta_j} y_i\right).$$

Since $\dim V > k$ and using the Theorem 3.2 we choose $u_1, \ldots, u_k$ such that

$$\bigwedge_{i=1}^{k} u_i \in \left\langle \bigwedge_{i\in\Delta_1} x_i \right\rangle^{\perp}.$$

Then,

$$\prod_{j=1}^{p}\left(\bigwedge_{i=1}^{k} u_i, \bigwedge_{i\in\Delta_j} y_i\right) = 0. \tag{8}$$

For $j = 1, \ldots, p$, let

$$\bigwedge_{i\in\Delta_j} y_i = v_j + w_j, \tag{9}$$

where $v_j \in \langle \bigwedge_{i\in\Delta_1} x_i \rangle$ and $w_j \in \langle \bigwedge_{i\in\Delta_1} x_i \rangle^{\perp}$.

From (8) and (9) we conclude that

$$\prod_{j=1}^{p}\left(\bigwedge_{i=1}^{k} u_i, \bigwedge_{i\in\Delta_j} y_i\right) = \prod_{j=1}^{p}\left(\bigwedge_{i=1}^{k} u_i, w_j\right) = 0.$$

Let $\mathscr{S}$ be the hyperplane of $\wedge^k V$

$$\mathscr{S} := \left\langle \bigwedge_{i\in\Delta_1} x_i \right\rangle^{\perp}.$$

For all $j = 1, \ldots, p$ let $\mathscr{P}_j$ be the hyperplane

$$\langle w_j\rangle^{\perp}.$$

Since

$$\prod_{j=1}^{p}\left(\bigwedge_{i=1}^{k} u_i, w_j\right) = 0$$

for all $u_1, \ldots, u_k$ such that

$$u_1 \wedge \cdots \wedge u_k \in \left\langle \bigwedge_{i\in\Delta_1} x_i \right\rangle^{\perp},$$

we can conclude

$$\mathscr{G} \cap \mathscr{S} \subseteq \mathscr{P}_1 \cup \cdots \cup \mathscr{P}_p.$$

So

$$\begin{aligned}\mathscr{G} \cap \mathscr{S} &= (\mathscr{G} \cap \mathscr{S}) \cap (\mathscr{P}_1 \cup \cdots \cup \mathscr{P}_p)\\ &= (\mathscr{G} \cap \mathscr{S} \cap \mathscr{P}_1) \cup \cdots \cup (\mathscr{G} \cap \mathscr{S} \cap \mathscr{P}_p).\end{aligned}$$

Since $\mathscr{G} \cap \mathscr{S}$ is an irreducible algebraic variety (Theorem 3.3), there exists $r \in \{1, \ldots, p\}$,

$$\mathscr{G} \cap \mathscr{S} = \mathscr{G} \cap \mathscr{S} \cap \mathscr{P}_r.$$

Then, $\mathscr{G} \cap \mathscr{S} \subseteq \mathscr{P}_r$. This inclusion implies that

$$(z, w_r) = 0$$

for all decomposable element $z \in \langle \bigwedge_{i\in\Delta_1} x_i\rangle^{\perp}$. Then, for all $\alpha \in \mathfrak{A}$, we have $(e^{\wedge}_{\alpha}, w_r) = 0$.

Since $w_r \in \langle \bigwedge_{i\in\Delta_1} x_i\rangle^{\perp}$ and $(e^{\wedge}_{\alpha})_{\alpha\in\mathfrak{A}}$ is an orthogonal basis of decomposable tensors for $\langle \bigwedge_{i\in\Delta_1} x_i\rangle^{\perp}$, we conclude that

$$w_r = 0.$$

Therefore,

$$\bigwedge_{i\in\Delta_r} y_i = v_r \in \left\langle \bigwedge_{i\in\Delta_1} x_i \right\rangle.$$

Thus

$$\bigwedge_{i\in\Delta_r} y_i = d_1 \bigwedge_{i\in\Delta_1} x_i,$$

where $d_1 = \det M[y_1, \ldots, y_m|x_1, \ldots, x_m][\Delta_r|\Delta_1]$. Therefore

$$\langle x_i : i \in \Delta_1\rangle = \langle y_i : i \in \Delta_r\rangle.$$

Let $l \in \{2, \ldots, p-1\}$ and assume (induction hypothesis) that for all $j \in \{1, 2, \ldots, l\}$ there exists $r_j \in \{1, \ldots, p\} \setminus \{r_1, r_2, \ldots, r_{j-1}\}$ such that

$$\langle x_i : i \in \Delta_j\rangle = \langle y_i : i \in \Delta_{r_j}\rangle.$$

Then, there exist $d_2, \ldots, d_l$ such that

$$\bigwedge_{i\in\Delta_{r_j}} y_i = d_j \bigwedge_{i\in\Delta_j} x_i, \quad j = 2, \ldots, l,$$

with $d_j = \det M[y_1, \ldots, y_m|x_1, \ldots, x_m][\Delta_{r_j}|\Delta_j]$, $j = 1, \ldots, l$. Since for every family $(u_1, \ldots, u_k)$ linearly independent we have

$$\prod_{j=1}^{p}\left(\bigwedge_{i=1}^{k} u_i, \bigwedge_{i\in\Delta_j} x_i\right) = \prod_{j=1}^{p}\left(\bigwedge_{i=1}^{k} u_i, \bigwedge_{i\in\Delta_j} y_i\right),$$

we obtain

$$\begin{aligned}\left(\bigwedge_{i=1}^{k} u_i, \bigwedge_{i\in\Delta_1} x_i\right)\cdots\left(\bigwedge_{i=1}^{k} u_i, \bigwedge_{i\in\Delta_l} x_i\right)\Bigg(\prod_{j=l+1}^{p}\left(\bigwedge_{i=1}^{k} u_i, \bigwedge_{i\in\Delta_j} x_i\right)\\ - d_1\cdots d_l \prod_{\substack{j=1\\ j\notin\{r_1,\ldots,r_l\}}}^{p}\left(\bigwedge_{i=1}^{k} u_i, \bigwedge_{i\in\Delta_j} y_i\right)\Bigg) = 0. \qquad (10)\end{aligned}$$

For $j \in \{l+1, \ldots, p\}$ let

$$\bigwedge_{i\in\Delta_j} x_i = \sum_{\alpha\in Q_{k,n}} c_{j\alpha}\overline{e}^{\wedge}_{\alpha}$$

and for $j \in \{1, \ldots, p\} - \{r_1, \ldots, r_l\}$ let

$$\bigwedge_{i\in\Delta_j} y_i = \sum_{\alpha\in Q_{k,n}} b_{j\alpha}\overline{e}^{\wedge}_{\alpha}.$$

Let $\mathscr{R}_j$ be the hyperplane

$$\left\langle \bigwedge_{i\in\Delta_j} x_i \right\rangle^{\perp}$$

and let $\mathscr{H}$ be the algebraic variety defined by the polynomial belonging to $\mathbb{C}[X_\alpha : \alpha \in Q_{k,n}]$

$$\prod_{j=l+1}^{p} \left(\sum_{\alpha\in Q_{k,n}} \overline{c_{j\alpha}} X_\alpha \right) - d_1 \cdots d_l \prod_{\substack{j=1 \\ j\notin\{r_1,\ldots,r_l\}}}^{p} \left(\sum_{\alpha\in Q_{k,n}} \overline{b_{j\alpha}} X_\alpha \right).$$

By (10) we conclude

$$\mathscr{G} \subseteq \mathscr{R}_1 \cup \cdots \cup \mathscr{R}_l \cup \mathscr{H}.$$

So

$$\mathscr{G} = \mathscr{G} \cap (\mathscr{R}_1 \cup \cdots \cup \mathscr{R}_l \cup \mathscr{H}) = (\mathscr{G} \cap \mathscr{R}_1) \cup \cdots \cup (\mathscr{G} \cap \mathscr{R}_l) \cup (\mathscr{G} \cap \mathscr{H}).$$

Since

$$\bigwedge_{i\in\Delta_j} x_i \neq 0,$$

we have

$$\left(\bigwedge_{i\in\Delta_j} x_i, \bigwedge_{i\in\Delta_j} x_i \right) \neq 0$$

and so,

$$\mathscr{G} \cap \mathscr{R}_j \neq \mathscr{G}, \; j = 1, \ldots, l.$$

Since $\mathscr{G}$ is an irreducible algebraic variety we get

$$\mathscr{G} = \mathscr{G} \cap \mathscr{H},$$

that is

$$\prod_{j=l+1}^{p} \left(\bigwedge_{i=1}^{k} u_i, \bigwedge_{i\in\Delta_j} x_i \right) = d_1 \cdots d_l \prod_{\substack{j=1 \\ j\notin\{r_1,\ldots,r_l\}}}^{p} \left(\bigwedge_{i=1}^{k} u_i, \bigwedge_{i\in\Delta_j} y_i \right)$$

for all linearly independent families $(u_1, \ldots, u_k)$ of vectors of V. Assume $u_1, \ldots, u_k$ are such that

$$\bigwedge_{i=1}^{k} u_i \in \left\langle \bigwedge_{i\in\Delta_{l+1}} x_i \right\rangle^{\perp}.$$

As before, we can conclude that there exists a $r_{l+1} \in \{1, 2, \dots, p\} - \{r_1, r_2, \dots, r_l\}$ such that

$$\langle x_i : i \in \Delta_{l+1} \rangle = \langle y_i : i \in \Delta_{r_{l+1}} \rangle.$$

So, by induction, there exists $\sigma \in S_p$ such that

$$\langle x_i : i \in \Delta_j \rangle = \langle y_i : i \in \Delta_{\sigma(j)} \rangle, \quad j = 1, \dots, p.$$

Denote by $s_{i,j}$ the integer in the box (i, j) of D and let $\sigma' \in R(D)$ such that

$$\sigma'(s_{i,j}) = s_{i,\sigma(j)}, \quad \text{for every } (i, j) \text{ box of } [\lambda].$$

Then, since $P(D)P(\sigma') = P(D)$,

$$\begin{aligned}
\frac{1}{|C|} E(D)(y_1 \otimes \cdots \otimes y_m) &= P(D)\left(\bigwedge_{i \in \Delta_1} y_i \otimes \cdots \otimes \bigwedge_{i \in \Delta_p} y_i \right) \\
&= P(D)P(\sigma')\left(\bigwedge_{i \in \Delta_{\sigma(1)}} y_i \otimes \cdots \otimes \bigwedge_{i \in \Delta_{\sigma(p)}} y_i \right) \\
&= \prod_{i=1}^{p} \det M[y_1, \dots, y_m | x_1, \dots, x_m][\Delta_{\sigma(i)} | \Delta_i] \\
&\quad \times P(D)\left(\left(\bigwedge_{i \in \Delta_1} x_i \right) \otimes \cdots \otimes \left(\bigwedge_{i \in \Delta_p} x_i \right) \right) \\
&= \prod_{i=1}^{p} \det M[y_1, \dots, y_m | x_1, \dots, x_m][\Delta_{\sigma(i)} | \Delta_i] \\
&\quad \times \frac{1}{|C|} E(D)(x_1 \otimes \cdots \otimes x_m).
\end{aligned}$$

From equality (6)

$$E(D)(x_1 \otimes \cdots \otimes x_m) = E(D)(y_1 \otimes \cdots \otimes y_m) \neq 0,$$

we obtain

$$\prod_{i=1}^{p} \det M[y_1, \dots, y_m | x_1, \dots, x_m][\Delta_{\sigma(i)} | \Delta_i] = 1.$$

Assume now that the conditions (1) and (2) are satisfied. To prove that

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m,$$

we start showing that

$$E(D)(x_1 \otimes \cdots \otimes x_m) = E(D)(y_1 \otimes \cdots \otimes y_m)$$

for all standard strict Young tableaux associated to $[\lambda]$.

Let D be a standard strict Young tableau associated to $[\lambda]$. If the columns of D are not a support of a λ'-coloring of $x_1, \ldots, x_m$ by (1) they are not a support of a λ'-coloring of $y_1, \ldots, y_m$ and then we have

$$E(D)(x_1 \otimes \cdots \otimes x_m) = E(D)(y_1 \otimes \cdots \otimes y_m) = 0.$$

Assume now that the columns of D, $\Delta_1, \ldots, \Delta_p$, are a support of a λ'-coloring of $x_1, \ldots, x_m$. Let

$$\Delta_i = \{s_{1,i}, \ldots, s_{k,i}\},\ s_{1,i} < \cdots < s_{k,i}, \quad i = 1, \ldots, p$$

and let θ be the permutation of S_m defined by the equalities

$$\theta(\lambda'_1 + \cdots + \lambda'_{j-1} + t) = s_{t,j},$$

$j = 1, \ldots, p$ and $t = 1, \ldots, k$.

Using as a model of the mth tensor power of V the pair $(\gamma, \otimes^m V)$, where $\gamma = P(\theta^{-1}) \circ \otimes$, the fact that there exists $\sigma \in S_p$ such

$$\langle x_i : i \in \Delta_j \rangle = \langle y_i : i \in \Delta_{\sigma(j)} \rangle, \quad j = 1, \ldots, p$$

and

$$\prod_{i=1}^{p} \det M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta_{\sigma(i)} | \Delta_i] = 1,$$

we have

$$\begin{aligned}
\frac{1}{|C|} E(D)(y_1 \otimes \cdots \otimes y_m) &= P(D)\left(\bigwedge_{i \in \Delta_1} y_i \otimes \cdots \otimes \bigwedge_{i \in \Delta_p} y_i \right) \\
&= P(D)P(\sigma')\left(\left(\bigwedge_{i \in \Delta_{\sigma(1)}} y_i \right) \otimes \cdots \otimes \left(\bigwedge_{i \in \Delta_{\sigma(p)}} y_i \right) \right) \\
&= \prod_{i=1}^{p} \det M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta_{\sigma(i)} | \Delta_i] \\
&\quad \times P(D)\left(\left(\bigwedge_{i \in \Delta_1} x_i \right) \otimes \cdots \otimes \left(\bigwedge_{i \in \Delta_p} x_i \right) \right) \\
&= \frac{1}{|C|} E(D)(x_1 \otimes \cdots \otimes x_m).
\end{aligned}$$

We conclude that

$$E(D)(x_1 \otimes \cdots \otimes x_m) = E(D)(y_1 \otimes \cdots \otimes y_m).$$

Now, bearing in mind Proposition 2.1, we get

$$e_{i,i}(x_1 \otimes \cdots \otimes x_m) = e_{i,i}(y_1 \otimes \cdots \otimes y_m), \quad i = 1, \ldots, d,$$

where d is the number of the standard strict Young tableaux associated with [λ], and so,

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m$$

and the proof is complete. □

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